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Functional Forms for the Squeeze and the Time-Displacement Operators

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ABSTRACT

Using Baker-Campbell-Hausdorff relations, the squeeze and harmonic-oscillator time-displacement operators are given in the form $\exp[\delta I]\exp[\alpha(x^2)]\exp[\beta(x\partial)]\exp[\gamma(\partial)^2]$, where α, β, γ, and δ are explicitly determined. Applications are discussed.

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1 Introduction

In group theory and quantum optics, the use of Baker-Campbell-Hausdorff relations [1, 2], to write unitary operators in more useful (often normal-ordered) forms, is a well-known technique. For instance, the displacement operator,

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] \ , \qquad \alpha = \alpha_1 + i\alpha_2 \equiv (x_0 + ip_0)/\sqrt{2} \ , \tag{1}$$

and the squeeze operator,

$$S(z) = \exp\left[\frac{1}{2} z a^\dagger a^\dagger - \frac{1}{2} z^* aa\right] \ , \qquad z = re^{i\phi} = z_1 + iz_2 \ , \tag{2}$$

can be written as

$$D(\alpha) = \exp\left[-\frac{1}{2}|\alpha|^2\right] \exp[\alpha a^\dagger] \exp[-\alpha^* a] \tag{3}$$

and

$$S(z) = \exp\left[\frac{1}{2} e^{i\theta} (\tanh r) a^\dagger a^\dagger\right] \left(\frac{1}{\cosh r}\right)^{(\frac{1}{2} + a^\dagger a)} \exp\left[-\frac{1}{2} e^{-i\theta} (\tanh r) aa\right] \tag{4}$$

$$= \exp\left[\frac{1}{2} e^{i\theta} (\tanh r) a^\dagger a^\dagger\right] (\cosh r)^{-1/2} \sum_{n=0}^{\infty} \frac{(\mathrm{sech} r - 1)^n}{n!} (a^\dagger)^n (a)^n$$

$$\times \ \exp\left[-\frac{1}{2} e^{-i\theta} (\tanh r) aa\right] \ . \tag{5}$$

However, such transformations are not as well-known in $x - p$ space, in terms of the variables

$$x = \frac{1}{\sqrt{2}} (a + a^\dagger) \ , \quad \partial = ip = \frac{1}{\sqrt{2}} (a - a^\dagger) \ , \tag{6}$$

$$[a, \ a^\dagger] = 1 \ , \qquad [x, \ \partial] = -1 \ . \tag{7}$$

Granted, the displacement operator is easily expressed as

$$D(\alpha) = \exp[-ix_0 p_0/2] \exp[ip_0 x] \exp[-x_0 \partial] \ . \tag{8}$$

but the squeeze operator

$$S(z) = \exp[-z_1(x\partial + 1/2) + iz_2(x^2 + \partial^2)/2] \ . \tag{9}$$

is not. If it were, it could be easily applied to any wave function, using the operator properties

$$\exp[c\partial]h(x) = h(x+c) \quad (10)$$

$$\exp[\tau(x\partial)]h(x) = h(xe^{\tau}) \quad (11)$$

$$\exp[c(\partial^2)]h(x) = \frac{1}{[4\pi c]^{1/2}}\int_{-\infty}^{\infty}\exp\left[-\frac{(y-x)^2}{4c}\right]h(y)dy\ . \quad (12)$$

In Section 2 we describe the BCH method to obtain such transformations. In Section 3 we explicitly apply it to the squeeze operator, and in Section 4 we do the same for the unitary, harmonic-oscillator, time-displacement operator

$$T = \exp[-i(a^{\dagger}a + 1/2)] = \exp[-i(x^2+\partial^2)/2]\ . \quad (13)$$

In Section 5 we give examples and indicate further work.

2 The Method

There are a number of early papers that deal with this subject, but a complete description, that we follow, was given by Wei and Norman [3]. It has been used widely. (See, e.g., Refs. [4, 5].) Consider a unitary operator $U(t)$ in terms of the exponentiations of I, x^2, $(x\partial)$, and $(\partial)^2$ times a parameter, t:

$$U(t) = \exp[t\{b_1 I + b_2 x^2 + b_3(x\partial) + b_4(\partial^2)\}]\ , \quad (14)$$

where the b_i are not functions of x or ∂. Set this equal to the following ordered product:

$$U_1(t) = \exp[\delta I]\exp[i\alpha x^2]\exp[\beta(x\partial)]\exp[i\gamma(\partial)^2]\ , \quad (15)$$

where $\alpha(t)$ (not to be confused with the coherent state label α), $\beta(t)$, $\gamma(t)$, and $\delta(t)$ are the functions to be determined. (Note that even in the case where U contains only I, x^2, and $(\partial)^2$ terms, one still needs a $(x\partial)$ term on the right since this operator is needed to close the algebra.)

Take the time derivative of both sides of the equality $U = U_1$, and then multiply on the right by $U^\dagger$ and $U_1^\dagger$, respectively. This yields

$$\begin{aligned} b_1 I + b_2 x^2 &+ b_3(x\partial) + b_4(\partial^2) \\ &= [e^{2\delta} e^{-\beta}]\{i\dot{\alpha}x^2 + \dot{\beta}\exp[i\alpha x^2](x\partial)\exp[-i\alpha x^2] \\ &\quad + \dot{\gamma}\exp[i\alpha x^2]\exp[\beta(x\partial)](\partial^2)\exp[-\beta(x\partial)]\exp[-i\alpha x^2] + \dot{\delta}\}\ , \end{aligned} \quad (16)$$

where "dot" signifies $\frac{d}{dt}$. The factor $[e^{2\delta}e^{-\beta}]$ is unity by $UU^\dagger = 1$. Now the main line operators on the right-hand side of the equation should be commuted to the right.

First do this with

$$X = \exp[i\alpha x^2](x\partial)\exp[-i\alpha x^2]\ . \quad (17)$$

Using

$$e^A B e^{-A} = B + [A, B] + [A, [A, B]]/2 + \dots\ , \quad (18)$$

and

$$[x^2, x\partial] = -2x^2\ , \qquad [x^2, [x^2, x\partial]] = 0\ , \quad (19)$$

one has

$$X = (x\partial) - i2\alpha x^2\ . \quad (20)$$

Similarly one has

$$\begin{aligned} Y &= \exp[\beta(x\partial)](\partial^2)\exp[-\beta(x\partial)] \\ &= \partial^2 \sum_{n+o}^{\infty} \frac{(-2\beta)^{2n}}{n!} = \partial^2 e^{-2\beta}\ . \end{aligned} \quad (21)$$

This leaves to calculate

$$\begin{aligned} Z &= \exp[i\alpha x^2]\partial^2\exp[-i\alpha x^2] \\ &= -i2\alpha - i4\alpha(x\partial) - 4\alpha^2(x^2) + \partial^2\ , \end{aligned} \quad (22)$$

where the last line comes from direct differentiation.

Putting Eqs. (20), (21), and (22) into Eq. (16), one has an equation with coefficients multiplying I, x^2, $(x\partial)$, and ∂^2. Because these are independent variables, that

means we can write the coefficients multiplying each of these variables as a separate equation:

$$
\begin{aligned}
b_1 &= \dot{\gamma} e^{-2\beta}(2\alpha) + \dot{\delta} \ , && (23) \\
b_2 &= i\dot{\alpha} - i2\alpha\dot{\beta} + i\dot{\gamma} e^{-2\beta}(-4\alpha^2) \ , && (24) \\
b_3 &= \dot{\beta} + \dot{\gamma} e^{-2\beta}(4\alpha) \ , && (25) \\
b_4 &= i\dot{\gamma} e^{-2\beta} \ . && (26)
\end{aligned}
$$

These are four first-order differential equations in four unknowns. They should be solved subject to the boundary conditions that the $b_i(0) = 0$. Then set $t = 1$ and one has the unitary operator in product form. (For the time-displacement case, one lets t simply remain t, the time.)

3 The Squeeze Operator

For the squeeze operator one has, from Eq. (9),

$$
b_1 = -\frac{z_1}{2} \ , \quad b_2 = i\frac{z_2}{2} \ , \quad b_3 = -z_1 \ , \quad b_4 = i\frac{z_2}{2} \ . \tag{27}
$$

Put this into Eqs. (23)-(26). With linear combinations of Eqs. (24) to (26) one finds

$$
\dot{\alpha} = -2z_2\alpha^2 - 2z_1\alpha + z_2/2 \ . \tag{28}
$$

The solution that goes to zero at $t = 0$ is

$$
\alpha(t) = \frac{z^2}{2r}\frac{\sinh rt}{\mathcal{S}(t)} \ , \tag{29}
$$

where

$$
\mathcal{S}(t) = \cosh rt + \frac{z_1}{r}\sinh rt \ . \tag{30}
$$

Putting this result into a linear combination of Eqs. (25) and (26),

$$
\dot{\beta} = -z_1 - 2z_2\alpha \ , \tag{31}
$$

or

$$\beta = -\ln \mathcal{S}(t) \ . \tag{32}$$

Now Eq. (26) gives us

$$\dot{\gamma} = \frac{z_2}{2} e^{2\beta} \ , \tag{33}$$

or

$$\gamma = \frac{z^2}{2r} \frac{\sinh rt}{\mathcal{S}(t)} = \alpha \ . \tag{34}$$

Finally, Eq. (23) gives us

$$\dot{\delta} = -\frac{z_1}{2} - 2\alpha\dot{\gamma}e^{2\beta} \ , \tag{35}$$

or

$$\delta = -\frac{1}{2}\ln \mathcal{S}(t) = \frac{\beta}{2} \ . \tag{36}$$

Therefore, setting $t = 1$, we obtain the squeeze operator,

$$S = \mathcal{S}^{-1/2} \exp\left[\frac{iz_2}{2r}\frac{\sinh r}{\mathcal{S}}(x^2)\right] \exp[-(\ln \mathcal{S})(x\partial)] \exp\left[\frac{iz_2}{2r}\frac{\sinh r}{\mathcal{S}}(\partial^2)\right] \ , \tag{37}$$

where

$$\mathcal{S} = \cosh r + \frac{z_1}{r}\sinh r = e^r \cos^2\frac{\phi}{2} + e^{-r}\sin^2\frac{\phi}{2} \ . \tag{38}$$

4 The Time-Displacement Operator

The harmonic-oscillator time-evolution operator can now be calculated in the same way, but with

$$b_1 = 0, \quad b_2 = -\frac{i}{2} \ , \quad b_3 = 0 \ , \quad b_4 = +\frac{i}{2} \ . \tag{39}$$

Then the solutions are found in the same way, this yielding, respectively,

$$\dot{\alpha} = -2\alpha^2 - 1/2 \ , \qquad \alpha(t) = -\frac{\tan t}{2} \ , \tag{40}$$

$$\alpha\dot{\beta} = 1/2 + \dot{\alpha} \ , \qquad \beta(t) = -\ln(\cos t) \ , \tag{41}$$

$$\dot{\gamma} = e^{2\beta}/2 \ , \qquad \gamma(t) = [\tan t]/2 \ , \tag{42}$$

$$\dot{\delta} = -\alpha \ , \qquad \delta(t) = -[\ln(\cos t)]/2 \ . \tag{43}$$

This means that the harmonic-oscillator time-displacement operator is

$$T = [\cos t]^{-1/2} \exp\left[-\frac{i}{2}\tan t(x^2)\right] \exp[-(\ln\cos t)(x\partial)] \exp\left[\frac{i}{2}\tan t(\partial^2)\right] \ . \qquad (44)$$

This result can be viewed as complementary to others [6] in the study of time-evolution.

5 Discussion

It is enlightening to look at specific examples.

Using Eqs. (8) and (38) on the harmonic-oscillator ground state,

$$\psi_0 = \pi^{-1/4} \exp[-x^2/2] \ , \qquad (45)$$

one finds

$$\begin{aligned} \psi_{ss} &= D(\alpha)S(z)\psi_0 \\ &= \frac{1}{\pi^{1/4}} \frac{\exp[-ix_0p_0/2]}{[\mathcal{S}(1+i2\kappa)]^{1/2}} \exp\left[-(x-x_0)^2\left(\frac{1}{2\mathcal{S}^2(1+i2\kappa)} - i\kappa\right) + ip_0x\right] \ , \end{aligned} \qquad (46)$$

where

$$\kappa = \frac{z_2 \sinh r}{2rs} \ . \qquad (47)$$

This is the most general squeezed state. Setting z to be real and positive yields the most commonly studied example:

$$\psi_{ss} = [\pi^{1/2}s]^{-1/2} \exp\left[-\frac{(x-x_0)^2}{2s^2} - ip_0x\right] \ , \qquad (48)$$

where

$$s = e^r \ . \qquad (49)$$

As a test, the time-evolution operator can be applied to the coherent states, which are Eq. (48) with $s = 1$. Then one finds

$$\begin{aligned} T\psi_{cs} &= \frac{e^{-it/2}}{\pi^{1/4}} \exp\left[-\frac{1}{2}\{x-(x_0\cos t + p_0\sin t)\}^2\right] \exp[ix(p_0\cos t - x_0\sin t)] \\ &\quad \exp\left[-\frac{i}{2}(x_0\cos t + p_0\sin t)(p_0\cos t - x_0\sin t)\right] \ . \end{aligned} \qquad (50)$$

Of course, a simpler calculation is possible by replacing α with αe^{-it} in the series defining the coherent states as an infinite sum of number states.

But the time-evolution operator can be applied to more complicated systems, for instance, the even and odd states of the harmonic oscillator. There, starting from the z real and $p_0 = 0$ squeezed states one can calculate [7]

$$\psi_{s\pm}(x,t) = T\psi_{s\pm}(x) \ . \tag{51}$$

One finds a closed form expression for $\psi_{s\pm}$:

$$\begin{aligned}
\psi_{s\pm}(x,t) \quad &= \quad \left[\frac{s}{2\pi^{1/2}(1 \pm e^{-x_0^2\cos^2 t})}\frac{s^2\cos t - i\sin t}{s^4\cos^2 t + \sin^2 t}\right]^{1/2} \\
&\left\{\exp\left[-\frac{(x - x_0\cos t)^2}{2}\left(\frac{s^2 - i\tan t}{s^4\cos^2 t + \sin^2 t}\right) - \frac{i}{2}(\tan t)x^2\right]\right. \\
&\left.\pm\exp\left[-\frac{(x + x_0\cos t)^2}{2}\left(\frac{s^2 - i\tan t}{s^4\cos^2 t + \sin^2 t}\right) - \frac{i}{2}(\tan t)x^2\right]\right\} . \qquad (52)
\end{aligned}$$

(Observe that the terms $\exp[-i(\tan t)x^2/2]$ are necessary to cancel the singularities of the terms $\exp[ix^2\tan t/(2\sin^2 t)]$ when t is an odd multiple of $\pi/2$.) This then yields an analytic description of the probability densities as a function of x, t:

$$\begin{aligned}
\rho_{s\pm}(x,t) \quad &= \quad \frac{\exp[-(x^2 + x_0^2\cos^2 t)/d^2]}{\pi^{1/2}d[1 \pm d\exp[-x_0^2/s^2]]} \\
&\left\{\cosh\left(\frac{2xx_0(\cos t)}{d^2}\right) \pm \cos\left(\frac{2xx_0\sin t}{d^2 s^2}\right)\right\} \ , \qquad (53)
\end{aligned}$$

where

$$d^2 = s^2\cos^2 t + \sin^2 t/s^2 \ . \tag{54}$$

Finally, it is amusing to apply the simple time-evolution operator for a particle in a box,

$$T_0 = \exp[+i(\partial)^2/2] \ , \tag{55}$$

to a number eigenstate of a particle in a box. One finds

$$T_0\sin(\pi n x) = \sin(\pi n x)\exp[-i\pi^2 n^2/2] \ , \tag{56}$$

the correct time-evolution of a number state. Here the operator is repeating the calculation for a continuous set of boxes along the real axis.

These techniques can be applied elsewhere, such as in obtaining the time-evolution operator for a system with different potentials. For example, the time-dependent system [8, 9]

$$V(x,t) = g^{(2)}(t)x^2 + g^{(1)}(t)x + g^{(0)}(t) \tag{57}$$

can be studied.

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